

MEDIANS, CONTINUITY, AND VANISHING OSCILLATION

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ABSTRACT. In this paper we consider properties of medians as they pertain to the continuity and vanishing oscillation of a function. Our approach is based on the observation that medians are related to local sharp maximal functions restricted to a cube of \mathbb{R}^n .

In considering the problem of the resistance of materials to certain types of deformations, F. John was led to the study of quasi-isometric mappings. The setting is essentially as follows. Let $Q_0 \subset \mathbb{R}^n$ be a cube and f a continuous function on Q_0 . Assume that to each subcube Q of Q_0 with sides parallel to those of Q_0 there is assigned a constant c_Q and let μ_Q be the function of the real variable M given by

$$\mu_Q(M) = \frac{|\{y \in Q : |f(y) - c_Q| > M\}|}{|Q|}.$$

Let $\phi(M) = \sup_{Q \subset Q_0} \mu_Q(M)$, $0 < s < 1/2$, and λ a number such that $\phi(\lambda) \leq s$. Then under these assumptions

$$\phi(M) \leq Ae^{-BM/\lambda}$$

holds for all nonnegative M where A, B are universal functions of s and the dimension n . Thus the space of functions of bounded mean oscillation (BMO) was introduced and the John-Nirenberg inequality established [5].

Strömberg adopted this setting when studying spaces close to BMO and considered different ways of describing the oscillation of a function f on cubes. He also incorporated the value $s = 1/2$ above, which corresponds to the notion of median value $m_f(Q) = m_f(1/2, Q)$ of f over Q [11]. The local maximal functions of Strömberg are of particular interest because they allow for pointwise estimates for Calderón-Zygmund singular integral operators [4, 6].

In this paper we consider properties of medians as they pertain to the continuity and vanishing oscillation of a function. Our approach is based on the observation that medians are related to local sharp maximal functions restricted to a cube $Q_0 \subset \mathbb{R}^n$ with parameter $0 < s \leq 1/2$ by means of the

2010 *Mathematics Subject Classification.* Primary 42B25; Secondary 46E30.

Key words and phrases. Medians, Local sharp maximal function.

expression

$$\begin{aligned} M_{0,s,Q_0}^{\sharp,\phi} f(x) &= \sup_{x \in Q, Q \subset Q_0} \inf_c \frac{m_{|f-c|}(1-s, Q)}{\phi(|Q|)} \\ &\sim \sup_{x \in Q, Q \subset Q_0} \frac{m_{|f-m_f(1-s, Q)|}(1-s, Q)}{\phi(|Q|)}, \end{aligned}$$

where $\phi = 1$ in the case of functions with bounded median oscillation with parameter s (bmo_s), and satisfies appropriate conditions in the case of functions with vanishing median oscillation with parameter s (vmo_s). Strömberg showed that $bmo_s = BMO$, and, similarly, we show here that for sufficiently small s , $vmo_s = VMO$, the space of functions of vanishing mean oscillation. Moreover, since VMO is known to contain bounded discontinuous functions [10, 8], we complete the picture by giving criteria for continuity for functions equivalent to a bounded function on a cube in terms of medians.

The paper is organized as follows. In Section 1 we introduce the notions of median and maximal median with respect to a parameter $0 < s < 1$; when $s \neq 1/2$ we refer to these medians as biased with parameter s . In Section 2 we consider the a.e. convergence of maximal biased medians in the spirit of Fujii's results for $s = 1/2$ [3]. In Section 3, motivated by similar results involving averages [9], we characterize continuity in terms of maximal biased medians with parameter $> 1/2$. In Section 4 we extend the Strömberg decomposition of cubes to parameters $> 1/2$. Finally, in Section 5 we consider the spaces of functions with vanishing median oscillation, establish a John-Nirenberg type inequality they satisfy, and show that, as anticipated, they coincide with VMO for sufficiently small s .

1. MEDIAN AND MAXIMAL MEDIAN

In what follows we restrict our attention to cubes with sides parallel to the coordinate axis.

Definition 1.1. For a cube $Q \subset \mathbb{R}^n$, $0 < s < 1$, and a real-valued measurable function f on Q we say that $m_f(s, Q)$ is a *median value of f over Q with parameter s* if

$$(1.1) \quad |\{y \in Q : f(y) < m_f(s, Q)\}| \leq s|Q|$$

and

$$(1.2) \quad |\{y \in Q : f(y) > m_f(s, Q)\}| \leq (1-s)|Q|.$$

When $s = 1/2$, $m_f(1/2, Q) = m_f(Q)$ corresponds to a median value of f over Q . The set of median values of f is one point or a closed interval as the example $f = \chi_{[1/2, 1]}$ on $[0, 1]$ shows. It is therefore convenient to work

with maximal medians, which are uniquely defined [1, 3]. More precisely, we have

Definition 1.2. For a cube $Q \subset \mathbb{R}^n$, $0 < s < 1$, and a real-valued measurable function f on Q , we say that $M_f(s, Q)$ is the *maximal median of f over Q with parameter s* if

$$M_f(s, Q) = \sup\{M : |\{y \in Q : f(y) < M\}| \leq s|Q|\}.$$

The reader will have no difficulty in proving the sup above is assumed, that is to say,

$$|\{y \in Q : f(y) < M_f(s, Q)\}| \leq s|Q|.$$

To justify the nomenclature of maximal median we verify that $M_f(s, Q)$ satisfies the conditions that characterize medians. (1.1) is guaranteed since $M_f(s, Q)$ is the maximum value for which it holds. As for (1.2), let $B_n = \{y \in Q : f(y) \geq M_f(s, Q) + 1/n\}$ and note that $|B_n| \leq (1 - s)|Q|$, all n , and $\{y \in Q : f(y) > M_f(s, Q)\} \subset \liminf_n B_n$. Then $|\{y \in Q : f(y) > M_f(s, Q)\}| \leq \liminf_n |B_n| \leq (1 - s)|Q|$.

Hereafter when considering a median we mean the maximal median and denote it simply by $m_f(s, Q)$. Clearly maximal medians satisfy

$$(1.3) \quad |\{y \in Q : f(y) \leq m_f(s, Q)\}| \geq s|Q|,$$

and

$$(1.4) \quad |\{y \in Q : f(y) \geq m_f(s, Q)\}| \geq (1 - s)|Q|.$$

We summarize the basic properties of maximal medians that are of interest to us in the following Proposition.

Proposition 1.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s, t < 1$, and f, g real-valued measurable function on Q . Then the following properties hold:*

(i) *For $s < t$,*

$$(1.5) \quad m_f(s, Q) \leq m_f(t, Q).$$

(ii) *If $f \leq g$ a.e., then*

$$(1.6) \quad m_f(s, Q) \leq m_g(s, Q).$$

(iii) *For a constant c ,*

$$(1.7) \quad m_f(s, Q) - c = m_{f-c}(s, Q).$$

(iv) *If $f, g \geq 0$ a.e., $0 < s, s_1 < 1$, and $0 < t < s + s_1 - 1$,*

$$(1.8) \quad m_{f+g}(t, Q) \leq m_f(s, Q) + m_g(s_1, Q).$$

(v) In general, $m_{-|f|}(s, Q) \leq m_f(s, Q) \leq m_{|f|}(s, Q)$. And, if $m_f(s, Q) \leq 0$,

$$(1.9) \quad |m_f(s, Q)| \leq m_{|f|}(1 - s, Q).$$

Thus for general f ,

$$(1.10) \quad |m_f(s, Q)| \leq m_{|f|}(s, Q), \quad 1/2 \leq s < 1.$$

(vi) If $f \geq 0$ is locally integrable and f_Q denotes the average of f over Q , then

$$(1.11) \quad m_f(s, Q) \leq \frac{1}{1-s} f_Q.$$

Proof. (i) Since $|\{y \in Q : f(y) < m_f(s, Q)\}| \leq s|Q| < t|Q|$, (1.5) holds.

(ii) Up to a set of measure zero $\{y \in Q : g(y) < m_f(s, Q)\} \subset \{y \in Q : f(y) < m_f(s, Q)\}$. Therefore $|\{y \in Q : g(y) < m_f(s, Q)\}| \leq s|Q|$, and so $m_f(s, Q) \leq m_g(s, Q)$.

(iii) Since $\{y \in Q : f(y) < m_f(s, Q)\} = \{y \in Q : f(y) - c < m_f(s, Q) - c\}$ it readily follows that $m_f(s, Q) - c \leq m_{f-c}(s, Q)$. And since $\{y \in Q : f(y) - c < m_{f-c}(s, Q)\} = \{y \in Q : f(y) < m_{f-c}(s, Q) + c\}$, $m_{f-c}(s, Q) + c \leq m_f(s, Q)$. Note that, in particular, $m_c(s, Q) = c$.

(iv) For the sake of argument suppose that f, g are measurable functions on Q such that $m_{f+g}(t, Q) - (m_f(s, Q) + m_g(s_1, Q)) > 2\eta > 0$. Then $\{y \in Q : f(y) < m_f(s, Q) + \eta\} \cap \{y \in Q : g(y) < m_g(s_1, Q) + \eta\} \subset \{y \in Q : f(y) + g(y) < m_{f+g}(t, Q)\}$. Now, since $|\{y \in Q : f(y) < m_f(s, Q) + \eta\}| \geq s|Q|$, $|\{y \in Q : g(y) < m_g(s_1, Q) + \eta\}| \geq s_1|Q|$, and $|\{y \in Q : f(y) + g(y) < m_{f+g}(t, Q)\}| \leq t|Q|$, it readily follows that $s + s_1 \leq 1 + t$, which is not the case.

(v) Since $-|f| \leq f \leq |f|$, by (1.6), $m_{-|f|}(s, Q) \leq m_f(s, Q) \leq m_{|f|}(s, Q)$. Now, if $m_f(s, Q) \leq 0$ note that $\{y \in Q : |f(y)| < -m_f(s, Q)\} = \{y \in Q : m_f(s, Q) < -|f(y)|\} \subset \{y \in Q : m_f(s, Q) < f(y)\}$, and, therefore, $|\{y \in Q : |f(y)| < -m_f(s, Q)\}| \leq (1-s)|Q|$. Consequently, $|m_f(s, Q)| \leq m_{|f|}(1-s, Q)$. And, since for $s \geq 1/2$, $(1-s) \leq s$, by (1.5) we have $|m_f(s, Q)| \leq m_{|f|}(s, Q)$ for that range of s .

(vi) We may assume that $m_f(s, Q) \neq 0$. Then by (1.4) and Chebychev's inequality,

$$(1-s)|Q| \leq |\{y \in Q : f(y) \geq m_f(s, Q)\}| \leq \frac{1}{m_f(s, Q)} \int_Q f(y) dy,$$

and the conclusion follows. \square

The restriction $1/2 \leq s < 1$ is necessary for (1.10) to hold. Let $Q = [0, 1]$ and $f(x) = -2\chi_{[0, 1/2)}(x) + \chi_{[1/2, 1]}(x)$; then for $0 < s < 1/2$, $m_f(s, Q) = -2$

but $m_{|f|}(s, Q) = 1 < 2$. And, in contrast to averages, the restriction $0 < t < s + s_1 - 1$ is necessary for (1.8) to hold. To see this let $Q = [0, 1]$, and pick $1/2 < s_1 \leq s < 1$, and $t = s + s_1 - 1 > 0$. If $f = \chi_{[0, 1-s]}$ and $g = \chi_{[1-s_1, 2(1-s_1)]}$, $\{y \in Q : f(y) + g(y) < 1\} = (1 - s, 1 - s_1) \cup (2(1 - s_1), 1]$ has measure $(s_1 - s) + 1 - 2(1 - s_1) = t$, and, therefore, although $m_f(s, Q) = m_g(s_1, Q) = 0$, $m_{f+g}(t, Q) = 1$.

Finally, maximal medians can be expressed in terms of distribution functions or nonincreasing rearrangements. Recall that the nonincreasing rearrangement f^* of f at level $\lambda > 0$ is given by $f^*(\lambda) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \leq \lambda\}$ and satisfies

$$(1.12) \quad |\{y \in \mathbb{R}^n : |f(y)| > f^*(u)\}| \leq u, \quad u > 0.$$

We then have

Proposition 1.2. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s < 1$, and f a measurable function on Q . Then*

$$m_{|f|}(1 - s, Q) = \inf\{\alpha > 0 : |\{y \in Q : |f(y)| > \alpha\}| < s|Q|\} = (f\chi_Q)^*(s|Q|).$$

Proof. Let $\bar{\alpha} = \inf\{\alpha > 0 : |\{y \in Q : |f(y)| > \alpha\}| < s|Q|\}$. Then for all $\varepsilon > 0$ it readily follows that $|\{y \in Q : |f(y)| \leq \bar{\alpha} + \varepsilon\}| > (1 - s)|Q|$, which together with (1.3) implies $m_{|f|}(1 - s, Q) \leq \bar{\alpha} + \varepsilon$. Thus $m_{|f|}(1 - s, Q) \leq \bar{\alpha}$.

Next, by (1.12), $|\{y \in Q : |f(y)| > (f\chi_Q)^*(s|Q|)\}| \leq s|Q|$, which gives $\bar{\alpha} \leq (f\chi_Q)^*(s|Q|)$.

Finally, since for $\varepsilon > 0$, $|\{y \in Q : |f(y)| > (f\chi_Q)^*(s|Q|) - \varepsilon\}| > s|Q|$, by (1.2) it readily follows that $(f\chi_Q)^*(s|Q|) - \varepsilon < m_{|f|}(1 - s, Q)$, and, consequently, $(f\chi_Q)^*(s|Q|) \leq m_{|f|}(1 - s, Q)$. \square

Other equivalent expressions appearing in the literature include those in [2].

2. CONVERGENCE OF MEDIAN

The examples following Proposition 1.1 suggest that medians rely more heavily on the distribution of the values of f than do averages. On the other hand, averages and medians are not always at odds. In particular, using (1.11) the reader should have no difficulty in verifying that the following version of the Lebesgue differentiation theorem holds: If f is a locally integrable function on \mathbb{R}^n and $1/2 \leq s < 1$, then

$$\lim_{x \in Q, Q \rightarrow x} m_f(s, Q) = f(x)$$

at every Lebesgue point x of f .

Thus, in some sense $m_f(s, Q)$ is a good substitute for f_Q for small Q . In fact, a more careful argument gives that the biased maximal medians $m_f(s, Q)$ of an arbitrary measurable function f converge to f a.e., a fact observed by Fujii for the case $s = 1/2$ [3].

Theorem 2.1. *Let f be a real-valued, finite a.e. measurable function on \mathbb{R}^n , and $0 < s < 1$. Then*

$$(2.1) \quad \lim_{x \in Q, Q \rightarrow x} m_f(s, Q) = f(x) \text{ a.e.}$$

In particular, (2.1) holds at every point of continuity x of f .

Proof. For $k \geq 1$ and an integer j , let $E_{k,j} = \{x \in \mathbb{R}^n : (j-1)/2^k \leq f(x) < j/2^k\}$, $a_{k,j} = (j-1)/2^k$, and put $S_k(x) = \sum_{j=-\infty}^{\infty} a_{k,j} \chi_{E_{k,j}}(x)$. Note that since f is finite a.e., $\mathbb{R}^n = \bigcup_{k,j} E_{k,j}$ except possibly for a set of measure 0, and when $f(x)$ is finite we have $0 \leq f(x) - S_k(x) \leq 2^{-k}$, which gives $m_{S_k}(s, Q) \leq m_f(s, Q) \leq m_{S_k}(s, Q) + 2^{-k}$ for all cubes Q . Let $A_{k,j} = \{x \in E_{k,j} : x \text{ is a point of density for } E_{k,j}\}$, $A_k = \bigcup_{j=-\infty}^{\infty} A_{k,j}$. Since f is finite a.e., $|\mathbb{R}^n \setminus A_k| = 0$ for all k , and if $A = \bigcup_{k=1}^{\infty} A_k$, also $|\mathbb{R}^n \setminus A| = 0$.

We claim that the limit in question exists for $x \in A$. Given $\varepsilon > 0$, pick k such that $2^{-k+1} < \varepsilon$. Then $x \in A_{k,j}$ for some j , and

$$\lim_{x \in Q, Q \rightarrow x} \frac{|A_{k,j} \cap Q|}{|Q|} = 1.$$

Let $\delta = \max\{s, 1-s\}$ and note that for all cubes Q with small enough measure containing x ,

$$\frac{|A_{k,j} \cap Q|}{|Q|} > \delta.$$

We restrict our attention to such small cubes Q containing x . Note that for these cubes $m_{S_k}(s, Q) = a_{k,j}$. Indeed, on the one hand, since $S_k(y) = a_{k,j}$ for $y \in A_{k,j}$, $|\{y \in Q : S_k(y) < a_{k,j}\}| \leq |A_{k,j}^c \cap Q| < s|Q|$, and, therefore, $a_{k,j} \leq m_{S_k}(s, Q)$. And, on the other, since for $\varepsilon > 0$, $\{y \in Q : S_k(y) < a_{k,j} + \varepsilon\} \supset A_{k,j} \cap Q$, it follows that $|\{y \in Q : S_k(y) < a_{k,j} + \varepsilon\}| \geq |A_{k,j} \cap Q| \geq (s + \eta)|Q|$. Hence, $m_{S_k}(s, Q) \leq a_{k,j} + \varepsilon$, and since ε is arbitrary, $m_{S_k}(s, Q) \leq a_{k,j}$.

Then, since $a_{k,j} = m_{S_k}(s, Q) = S_k(x)$ for $x \in A_{k,j}$,

$$\begin{aligned} |m_f(s, Q) - f(x)| &\leq |m_f(s, Q) - m_{S_k}(s, Q)| + |m_{S_k}(s, Q) - f(x)| \\ &\leq 2^{-k} + (f(x) - S_k(x)) \leq 2^{-k+1} < \varepsilon. \end{aligned}$$

In other words, $|m_f(s, Q) - f(x)| < \varepsilon$ for $x \in A$ and all Q with small enough measure containing x .

Now, at a point of continuity x of f , given $\varepsilon > 0$, let $\delta > 0$ be such that $|f(y) - f(x)| \leq \varepsilon$ for $y \in B(x, \delta)$. Then for y in a cube Q containing x and contained in $B(x, \delta)$ we have $-\varepsilon \leq f(y) - f(x) \leq \varepsilon$, and, consequently, $-\varepsilon = m_{-\varepsilon}(s, Q) \leq m_{f-f(x)}(s, Q) = m_f(s, Q) - f(x) \leq m_\varepsilon(s, Q) = \varepsilon$, and so $|m_f(s, Q) - f(x)| \leq \varepsilon$. \square

3. A MEDIAN CHARACTERIZATION FOR CONTINUITY

We say that a measurable function f on a cube $Q_0 \subset \mathbb{R}^n$ is equivalent to a continuous function on Q_0 if the values of f can be modified on a set of Lebesgue measure 0 so as to coincide with a continuous function on Q_0 ; similarly for f equivalent to a bounded function on a cube. In this section we characterize those measurable functions equivalent to a bounded function on a cube that are equivalent to a continuous function on that cube in terms of medians, keeping in mind that in the case of locally integrable functions the condition involves the consideration of oscillations involving two nonoverlapping cubes [9].

Definition 3.1. For $0 < s < 1$ and nonoverlapping cubes $Q_1, Q_2 \subset \mathbb{R}^n$, let

$$\Psi_s(f, Q_1, Q_2) = \frac{|Q_1|}{|Q_1 \cup Q_2|} m_f(s, Q_1) + \frac{|Q_2|}{|Q_1 \cup Q_2|} m_f(s, Q_2),$$

and

$$\Omega(f, s, \delta) = \sup_{\text{diam}(Q_1 \cup Q_2) \leq \delta} \inf_c \Psi_s(|f - c|, Q_1, Q_2).$$

Ψ_s is a weighted average of maximal medians of f in the spirit of averages and $\Omega(f, s, \delta)$ is related to the oscillation of a measurable function on a cube, as shown by the following result.

Theorem 3.1. Let $Q_0 \subset \mathbb{R}^n$ be a cube, $1/2 < s < 1$, f a measurable function on Q_0 that is equivalent to a bounded function there, and $\omega(f, \delta)$, $\delta > 0$, the essential modulus of continuity of f defined by

$$\omega(f, \delta) = \sup_{|h| \leq \delta} \left(\text{ess sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \right).$$

Then we have $\Omega(f, s, \delta) = \omega(f, \delta)/2$.

Proof. For nonoverlapping cubes $Q_1, Q_2 \subset Q_0$, let

$$\Theta = \text{ess osc}(f, Q_1 \cup Q_2) = \text{ess sup}_{Q_1 \cup Q_2} f - \text{ess inf}_{Q_1 \cup Q_2} f.$$

Let $\delta > 0$. If $x \in Q_1 \cup Q_2 \subset Q_0$ is such that $x+h \in Q_0$ where $|h| \leq \delta$, since

$$\text{ess sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \geq \text{ess sup}_{Q_1 \cup Q_2} f - \text{ess inf}_{Q_1 \cup Q_2} f,$$

taking the sup over $|h| \leq \delta$ it readily follows that $\omega(f, \delta) \geq \Theta$. Moreover, since for $y \in Q_1$ and an arbitrary constant c ,

$$|f(y) - c| \leq \max \{ \text{ess sup}_{Q_1 \cup Q_2} f - c, c - \text{ess inf}_{Q_1 \cup Q_2} f \},$$

picking $c = (\sup \inf_{Q_1 \cup Q_2} f + \text{ess inf}_{Q_1 \cup Q_2} f)/2$, it follows that $|f(y) - c| \leq \Theta/2$, and, consequently, $m_{|f-c|}(s, Q_1) \leq m_{\Theta/2}(s, Q_1) = \Theta/2$; similarly we have $m_{|f-c|}(s, Q_2) \leq \Theta/2$. Therefore,

$$\inf_c \Psi_s(|f - c|, Q_1, Q_2) \leq \frac{|Q_1|}{|Q_1 \cup Q_2|} \frac{\Theta}{2} + \frac{|Q_2|}{|Q_1 \cup Q_2|} \frac{\Theta}{2} = \frac{\Theta}{2},$$

and, consequently, $\Omega(f, s, \delta) \leq \Theta/2 \leq \omega(f, \delta)/2$.

Conversely, let $0 < t < 2s - 1$. Then for fixed $\delta > 0$, given $\varepsilon > 0$, pick h with $|h| < \delta$ such that $\text{ess sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \geq \omega(f, \delta) - \varepsilon$. Then $E = \{x \in Q_0 : x+h \in Q_0 \text{ and } |f(x+h) - f(x)| \geq \omega(f, \delta) - \varepsilon\}$ has positive measure. Let $x \in E$ be a point of density of E and a small enough so that $Q(x, a), Q(x+h, a)$ are nonoverlapping and

$$\frac{|E \cap Q(x, a)|}{|Q(x, a)|} > 1 - t.$$

Now, since $\{y \in Q(x+h, a) : g(y) < M\} = \{y \in Q(x, a) : g(y+h) < M\}$ and $|Q(x, a)| = |Q(x+h, a)|$, it readily follows that $m_{|f-c|}(s, Q(x+h, a)) = m_{|f(\cdot+h)-c|}(s, Q(x, a))$, and, consequently, since $|f(y+h) - f(y)| \leq |f(y+h) - c| + |f(y) - c|$, by (1.8) and (1.6), $m_{|f-c|}(s, Q(x, a)) + m_{|f-c|}(s, Q(x+h, a)) \geq m_{|f-c|+|f(\cdot+h)-c|}(t, Q(x, a)) \geq m_{|f(\cdot+h)-f|}(t, Q(x, a))$. Therefore,

$$\begin{aligned} \Psi_s(|f-c|, Q(x, a), Q(x+h, a)) \\ \geq \frac{1}{2} m_{|f-c|}(s, Q(x, a)) + \frac{1}{2} m_{|f-c|}(s, Q(x+h, a)) \\ \geq \frac{1}{2} m_{|f(\cdot+h)-f|}(t, Q(x, a)). \end{aligned}$$

Finally, since $|E \cap Q(x, a)| = |\{y \in Q(x, a) : |f(y+h) - f(y)| \geq \omega(f, \delta) - \varepsilon\}| > (1-t)|Q(x, a)|$, it readily follows that $m_{|f(\cdot+h)-f|}(t, Q(x, a)) \geq \omega(f, \delta) - \varepsilon$, which, since ε is arbitrary, implies $\Psi_s(|f-c|, Q(x, a), Q(x+h, a)) \geq \omega(f, \delta)/2$. Thus $\Omega(f, s, \delta + (\sqrt{2a})^n) \geq \omega(f, \delta)$, and letting $a \rightarrow 0$, $\Omega(f, s, \delta) \geq \omega(f, \delta)/2$. \square

Theorem 3.2. *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $1/2 < s < 1$, and f a measurable function on Q_0 that is equivalent to a bounded function there. Then, f is equivalent to a continuous function on Q_0 iff $\lim_{\eta \rightarrow 0^+} \Omega(f, s, \eta) = 0$.*

The proof follows at once from Theorem 3.1. Note that by Proposition 1.2 the conclusion can also be stated in terms of rearrangements.

4. A DECOMPOSITION OF CUBES

Strömberg's essential tool in dealing with the oscillation of functions and local maximal functions is a decomposition of cubes [11]. In this section we extend the results to biased medians with parameters $> 1/2$.

We begin by introducing the local sharp maximal function restricted to a cube.

Definition 4.1. Let $Q_0 \subset \mathbb{R}^n$ and $0 < s \leq 1/2$. For a measurable function f on Q_0 , $M_{0,s,Q_0}^\sharp f(x)$, the *local sharp maximal function restricted to Q_0* of f is defined at $x \in Q_0$ as

$$(4.1) \quad M_{0,s,Q_0}^\sharp f(x) = \sup_{x \in Q, Q \subset Q_0} \inf_c \inf \{ \alpha \geq 0 : |\{y \in Q : |f(y) - c| > \alpha\}| < s|Q| \}.$$

When $Q_0 = \mathbb{R}^n$, $M_{0,s,\mathbb{R}^n}^\sharp f(x) = M_{0,s}^\sharp f(x)$ denotes the *local sharp maximal function of f at $x \in \mathbb{R}^n$* .

The range $0 < s \leq 1/2$ is necessary since for $s > 1/2$, $M_{0,s,Q}^\sharp f(x) = 0$ for a function f that takes two different values.

Local maximal functions, as well as maximal functions defined in terms in rearrangements, can be expressed in terms of medians. Let $\omega_s(f, Q) = \inf_c ((f - c)\chi_Q)^*(s|Q|)$. Then by Proposition 1.2,

$$M_{0,s,Q_0}^\sharp f(x) = \sup_{x \in Q, Q \subset Q_0} \omega_s(f, Q) = \sup_{x \in Q, Q \subset Q_0} \inf_c m_{|f-c|}(1-s, Q).$$

The first expression above is used by Lerner [6, 7].

An efficient choice for c in the infimum above is $m_{|f-m_f(1-s,Q)|}(1-s, Q)$. Indeed, for $Q \subset Q_0$ and a constant c , since $1-s \geq 1/2$, by (1.7) and (1.10),

$$(4.2) \quad |m_f(1-s, Q) - c| \leq m_{|f-c|}(1-s, Q).$$

Then, since $|f(y) - m_f(1-s, Q)| \leq |f(y) - c| + |c - m_f(1-s, Q)|$, by (1.5), (1.7), and (4.2),

$$\begin{aligned} m_{|f-m_f(1-s,Q)|}(1-s, Q) &\leq m_{|f-c|}(1-s, Q) + |c - m_f(1-s, Q)| \\ &\leq m_{|f-c|}(1-s, Q) + m_{|f-c|}(1-s, Q) = 2 m_{|f-c|}(1-s, Q), \end{aligned}$$

and, consequently,

$$(4.3) \quad \inf_c m_{|f-c|}(1-s, Q) \leq m_{|f-m_f(1-s,Q)|}(1-s, Q) \leq 2 \inf_c m_{|f-c|}(1-s, Q).$$

The decomposition of cubes relies on three lemmas which we prove next.

Lemma 4.1. Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \leq 1/2$, $1/2 \leq t \leq 1-s$, and f a measurable function on Q . Then for any $\eta > 0$,

$$(4.4) \quad |\{y \in Q : |f(y) - m_f(t, Q)| \geq 2 \inf_{x \in Q} M_{0,s,Q}^\sharp f(x) + \eta\}| < s|Q|.$$

Proof. For fixed c , let $\alpha(c) = m_{|f-c|}(1-s, Q)$. Then by (4.2) and (1.5),

$$(4.5) \quad |m_f(t, Q) - c| \leq m_{|f-c|}(t, Q) \leq m_{|f-c|}(1-s, Q) = \alpha(c),$$

and by (1.4)

$$(4.6) \quad |\{y \in Q : |f(y) - c| \geq \alpha(c) + \varepsilon\}| < s|Q|, \quad \varepsilon > 0.$$

Let $m = \inf_c \alpha(c)$ and pick $\{c_k\}$ such that $m \leq \alpha(c_k) \leq m + 1/k$, all k . Then by (4.5),

$$\begin{aligned} |f(y) - c_k| &\geq |f(y) - m_f(t, Q)| - |m_f(t, Q) - c_k| \\ &\geq |f(y) - m_f(t, Q)| - \alpha(c_k), \end{aligned}$$

and, consequently, since $2m + \eta \geq \alpha(c_k) + (\eta - 2/k)$, $\{y \in Q : |f(y) - m_f(t, Q)| \geq 2m + \eta\} \subset \{y \in Q : |f(y) - c_k| \geq \alpha(c_k) + \varepsilon_k\}$, where we have chosen k sufficiently large so that $\varepsilon_k = \eta - (2/k) > 0$. Then by (4.6), $|\{y \in Q : |f(y) - m_f(t, Q)| > 2m + \eta\}| < s|Q|$. Finally, since $M_{0,s,Q}^\# f(x) \geq m$ for all $x \in Q$, (4.4) holds. \square

Lemma 4.2. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \leq 1/2$, $1/2 \leq t \leq 1-s$, $\eta > 0$, and f a measurable function on an open cube containing Q . Then for any family of cubes $\{Q_\varepsilon\}$ with $(1-\varepsilon)Q \subset Q_\varepsilon \subset (1+\varepsilon)Q$,*

$$\limsup_{\varepsilon \rightarrow 0^+} |m_f(t, Q) - m_f(t, Q_\varepsilon)| \leq 2 \inf_{x \in Q} M_{0,s,Q}^\# f(x) + \eta.$$

Proof. Let $A = \inf_{x \in Q} M_{0,s,Q}^\# f(x)$. For the sake of argument assume there is a sequence $\varepsilon_k \rightarrow 0$ such that $|m_f(t, Q) - m_f(t, Q_{\varepsilon_k})| > 2A + \eta$ for all k . Then by (1.10),

$$2A + \eta < |m_f(t, Q) - m_f(t, Q_{\varepsilon_k})| \leq m_{|f-m_f(t,Q)|}(t, Q_{\varepsilon_k}),$$

and, consequently, by (1.4),

$$(4.7) \quad |\{y \in Q_{\varepsilon_k} : |f(y) - m_f(t, Q)| > 2A + \eta\}| \geq (1-t)|Q_{\varepsilon_k}|.$$

Since $Q_{\varepsilon_k} \subset (1+\varepsilon_k)Q$, the left-hand side of (4.7) is bounded above by

$$\begin{aligned} &|\{y \in (1+\varepsilon_k)Q : |f(y) - m_f(t, Q)| > 2A + \eta\}| \\ &\leq ((1+\varepsilon_k)^n - 1)|Q| + |\{y \in Q : |f(y) - m_f(t, Q)| > 2A + \eta\}|, \end{aligned}$$

and since $(1-\varepsilon_k)Q \subset Q_{\varepsilon_k}$, the right-hand side of (4.7) is bounded below by

$$(1-t)(1-\varepsilon_k)^n|Q|.$$

Whence combining these estimates it follows that

$$|\{y \in Q : |f(y) - m_f(t, Q)| > 2A + \eta\}| \geq ((1-t)(1-\varepsilon_k)^n - ((1+\varepsilon_k)^n - 1))|Q|.$$

Now, by (4.2) there exists $\delta > 0$ such that $|\{y \in Q : |f(y) - m_f(t, Q)| \geq 2A + \eta\}| = (s - \delta)|Q|$, and so

$$(s - \delta) \geq ((1 - t)(1 - \varepsilon_k)^n - ((1 + \varepsilon_k)^n - 1)).$$

Thus, letting $k \rightarrow \infty$, $s - \delta \geq 1 - t \geq s$, which is not the case. \square

Lemma 4.3. *Let $Q_0, Q_1 \subset \mathbb{R}^n$ be cubes with $Q_0 \subset Q_1$ and $|Q_1| \leq 2^k|Q_0|$ for some integer k , $0 < s \leq 1/2$, $1/2 \leq t \leq 1 - s$, and f a measurable function on Q_1 . Then*

$$(4.8) \quad |m_f(t, Q_0) - m_f(t, Q_1)| \leq 10k \inf_{x \in Q_0} M_{0,s,Q_1}^\# f(x).$$

Proof. By the triangle inequality it suffices to prove the case $k = 1$. Let $A = \inf_{x \in Q_0} M_{0,s,Q_1}^\# f(x)$. For the sake of argument suppose that (4.8) does not hold. Then if $A > 0$, by Lemma 4.2, for any fixed $0 < \eta < A/2$, there exists a cube Q_2 such that $Q_0 \subset Q_2 \subset Q_1$ and

$$|m_f(t, Q_2) - m_f(t, Q_0)| > 4A + 2\eta, \quad |m_f(t, Q_2) - m_f(t, Q_1)| > 4A + 2\eta.$$

And if $A = 0$, then $|m_f(t, Q_0) - m_f(t, Q_1)| > 0$ and there exists a cube Q_2 such that $Q_0 \subset Q_2 \subset Q_1$ and

$$|m_f(t, Q_2) - m_f(t, Q_0)| > 2\eta, \quad |m_f(t, Q_2) - m_f(t, Q_1)| > 2\eta$$

for η sufficiently small.

Thus in both cases the sets $\{y \in Q_k : |f(y) - m_f(t, Q_k)| \leq 2A + \eta\}$, $k = 0, 1, 2$, are pairwise disjoint subsets of Q_1 , and, consequently,

$$\begin{aligned} & \{y \in Q_0 : |f(y) - m_f(t, Q_0)| \leq 2A + \eta\} \\ & \cup \{y \in Q_2 : |f(y) - m_f(t, Q_2)| \leq 2A + \eta\} \\ & \subset \{y \in Q_1 : |f(y) - m_f(t, Q_1)| > 2A + \eta\}. \end{aligned}$$

Therefore, since $\inf_{x \in Q_k} M_{0,s,Q_k}^\# f(x) \leq A$ for $k = 0, 1, 2$, by Lemma 4.1,

$$(1 - s)|Q_0| + (1 - s)|Q_2| < s|Q_1|.$$

Thus $2(1 - s)|Q_0| < s|Q_1| < 2s|Q_0|$, and, consequently, $1 < 2s$, which is not the case. \square

We are now ready to consider the decomposition of cubes relative to medians.

Proposition 4.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \leq 1/2$, $1/2 \leq t \leq 1 - s$, $\delta, \beta > 0$, and f a measurable function on Q . Then if $|m_f(t, Q)| \leq \delta$, there exists a (possibly empty) family of nonoverlapping dyadic subcubes $\{Q_k\}$ of Q so that*

$$(1) \quad Q_k \not\subset \{y \in Q : M_{0,s,Q}^\# f(y) > \beta\},$$

- (2) $\delta < |m_f(t, Q_k)| \leq \delta + 10n\beta$,
 (3) $|f(x)| \leq \delta$ for a.e. $x \in Q \setminus (\{y \in Q : M_{0,s,Q}^\# f(y) > \beta\} \cup \bigcup_k Q_k)$.

Proof. If $M_{0,s,Q}^\# f(y) > \beta$ for all $y \in Q$ we pick $\{Q_k\}$ as the empty family. Otherwise subdivide Q dyadically into 2^n subcubes and note that by Lemma 4.3 for each dyadic subcube Q' , $|m_f(t, Q')| \leq \delta + 10n \inf_{x \in Q'} M_{0,s,Q}^\# f(x)$. Thus for each of these subcubes Q' , say, one of the following holds:

- (a) $Q' \subset \{y \in Q : M_{0,s,Q}^\# f(y) > \beta\}$: we discard Q' .
 (b) Q' satisfies conditions (1) and (2) above: we collect this Q' .
 (c) $Q' \not\subset \{y \in Q : M_{0,s,Q}^\# f(y) > \beta\}$ but $|m_f(t, Q')| \leq \delta$: we subdivide Q' and continue in this fashion.

Finally, a.e. $x \in Q \setminus (\{y \in Q : M_{0,s,Q}^\# f(y) > \beta\} \cup \bigcup_k Q_k)$ is contained in arbitrarily small cubes $\{Q_k(x)\}$ containing x so that $|m_f(t, Q_k(x))| \leq \delta$. By Theorem 2.1 it readily follows that $|f(x)| \leq \delta$ for a.e. such x . \square

We can be more precise in the description of the cubes above when $M_{0,s,Q}^\# f \in L^\infty(Q)$. Observe that $f - m_f(t, Q)$ satisfies $m_{f-m_f(t,Q)}(t, Q) = 0$ and $M_{0,s,Q}^\# f(x) = M_{0,s,Q}^\# (f - m_t(t, Q))(x)$ for all $x \in Q$, which means that the decomposition for $f - m_f(t, Q)$ holds for any $\delta > 0$. Let $\{Q_j\}$ and $\{Q_k\}$ denote the families of cubes obtained from the decomposition with parameters $\beta > \|M_{0,s,Q}^\# f\|_{L^\infty(Q)}$ and $\delta_1 = 4\beta + 2\eta$ and $\delta_2 = 2\delta_1 + 10n\beta$, respectively.

Observe that by construction we have $\bigcup_k Q_k \subset \bigcup_j Q_j$. To see this consider a dyadic subcube Q' of Q that has not been discarded; this depends on β and not on δ_1 or δ_2 . If Q' is a Q_j , then $\delta_1 < |m_{f-m_f(t,Q)}(t, Q')| \leq \delta_1 + 10n\beta < \delta_2$ and Q' is not a Q_k . So any Q_k contained in Q' arises from subsequent subdivisions of Q' . On the other hand, if Q' is not a Q_j , then $|m_{f-m_f(t,Q)}(t, Q')| \leq \delta_1 < \delta_2$ and Q' is not a Q_k either. Since this relation is maintained at every level of the successive dyadic subdivisions, the Q_k 's arise from subdivisions of Q_j 's.

From here on the argument proceeds as in Lemma 4.3. Note that

$$\delta_1 < |m_{f-m_f(t,Q)}(t, Q_j)| = |m_f(t, Q_j) - m_f(t, Q)| \leq \delta_1 + 10n\beta = \delta_2 - \delta_1,$$

and

$$\delta_2 < |m_{f-m_f(t,Q)}(t, Q_k)| = |m_f(t, Q) - m_f(t, Q_k)|.$$

Therefore,

$$\begin{aligned} |m_f(t, Q_j) - m_f(t, Q_k)| &\geq |m_f(t, Q_k) - m_f(t, Q)| - |m_f(t, Q_j) - m_f(t, Q)| \\ &\geq \delta_2 - (\delta_2 - \delta_1) = \delta_1 = 4\beta + 2\eta. \end{aligned}$$

Thus it readily follows that the sets $\{y \in Q : |f(y) - m_f(t, Q)| \leq 2\beta + \eta\}$, $\{y \in Q_j : |f(y) - m_f(t, Q_j)| \leq 2\beta + \eta\}$, and $\{y \in Q_k : |f(y) - m_f(t, Q_k)| \leq 2\beta + \eta\}$ are nonoverlapping, and so

$$(4.9) \quad \begin{aligned} & \{y \in Q_j : |f(y) - m_f(t, Q_j)| \leq 2\beta + \eta\} \\ & \cup \{y \in Q_k : |f(y) - m_f(t, Q_k)| \leq 2\beta + \eta\} \\ & \subset \{y \in Q : |f(y) - m_f(t, Q)| > 2\beta + \eta\}. \end{aligned}$$

Now, since $\inf_{x \in Q_j} M_{0,s,Q_j}^\# f(x) \leq \beta$ for each Q_j , by (4.4) it follows that

$$(1-s) \sum_j |Q_j| \leq \sum_j |\{y \in Q_j : |f(y) - m_f(t, Q_j)| \leq 2\beta + \eta\}|,$$

and a similar estimate holds with the Q_k 's in place of the Q_j 's. Finally, since the sets in the left-hand side of (4.9) are pairwise disjoint for all j and k , and since $\bigcup_k Q_k \subset \bigcup_j Q_j$ and $\inf_{x \in Q} M_{0,s,Q}^\# f(x) \leq \beta$, by Lemma 4.1,

$$2 \sum_k |Q_k| \leq \sum_k |Q_k| + \sum_j |Q_j| \leq \frac{s}{1-s} |Q|,$$

and, consequently, since $2(1-s) \geq 1$,

$$(4.10) \quad \sum_k |Q_k| \leq \frac{s}{2(1-s)} |Q| \leq s |Q|.$$

5. VANISHING MEDIAN OSCILLATION

We say that a measurable function f defined on a cube $Q_0 \subset \mathbb{R}^n$ is of vanishing median oscillation with parameter s in Q_0 ($vmo_s(Q_0)$) if

$$\phi_s(u) = \sup_{Q \subset Q_0, |Q| \leq u} \inf_c m_{|f-c|}(1-s, Q)$$

satisfies $\lim_{u \rightarrow 0^+} \phi_s(u) = 0$.

Note that by (1.11),

$$\inf_c m_{|f-c|}(1-s, Q) \leq \frac{1}{s} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy,$$

and, therefore, $\lim_{u \rightarrow 0^+} \phi_s(u) = 0$ for all s whenever $f \in VMO(Q_0)$. Here we show that the spaces actually coincide for $s \leq 2^{-n}$.

Now, ϕ_s is a nonnegative, nondecreasing continuous function that vanishes at the origin, and vmo_s may be described in terms of such functions as follows. Let

$$\|f\|_{s,\phi,Q_0} = \sup_{Q \subset Q_0} \inf_c \frac{m_{|f-c|}(1-s, Q)}{\phi(|Q|)} \sim \sup_{Q \subset Q_0} \frac{m_{|f-m_f(1-s,Q)|}(1-s, Q)}{\phi(|Q|)},$$

and $bmo_{s,\phi}(Q_0) = \{f : f \text{ is defined and measurable on } Q_0, \text{ and } \|f\|_{s,\phi,Q_0} < \infty\}$. Then $vmo_s(Q_0) = \bigcup_\phi bmo_{s,\phi}(Q_0)$.

Now fix Q_0 and $0 < s \leq 1/2^n$. Let $\phi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be continuous, nondecreasing, and $\phi(0) = 0$, and define $\Psi_{|Q_0|} : [0, 2^n|Q_0|] \rightarrow \mathbb{R}^+$ by

$$(5.1) \quad \Psi_{|Q_0|}(u) = \int_u^{2^n|Q_0|} \frac{\phi(v)}{v} dv.$$

We can then prove a strengthened version of the John-Nirenberg inequality.

Theorem 5.1. *Let $f \in bmo_{s,\phi}(Q_0)$ for some ϕ as above and $\Psi_{|Q_0|}(u)$ given by (5.1). Then there exist constants c_1, c_2 independent of f and all subcubes $Q \subset Q_0$ so that*

$$(5.2) \quad |\{y \in Q : |f(y) - m_f(1-s, Q)| > \lambda\}| \leq c_1 \Psi_{|Q|}^{-1}(c_2 \lambda / \|f\|_{s,\phi,Q}), \quad \lambda > 0.$$

Proof. If $\|f\|_{s,\phi,Q_0} = 0$, clearly $M_{0,s,Q_0}^\# f(x) = 0$ for all $x \in Q_0$ and by Lemma 4.3, the medians of f over all subcubes of Q_0 are constant. Then by Theorem 2.1, f is a.e. constant, and the conclusion holds in the case. Otherwise, since $\|f - c\|_{s,\phi,Q_0} = \|f\|_{s,\phi,Q_0}$ and $\|cf\|_{s,\phi,Q_0} = |c| \|f\|_{s,\phi,Q_0}$ for all constants c we may assume that $\|f\|_{s,\phi,Q_0} = 1$ and $m_f(1-s, Q_0) = 0$. Then, by (4.3)

$$m_{|f-m_f(1-s,Q)|}(t, Q) \leq 2\phi(|Q|) \leq 2\phi(|Q_0|) \quad \text{for all } Q \subset Q_0,$$

and, consequently, $\|M_{0,s,Q_0}^\# f\|_{L^\infty(Q_0)} \leq 2\phi(|Q_0|)$. Pick now $\beta_0 = 2\phi(|Q_0|)$, and note that since $\phi(u) > 0$ for $u > 0$, $\delta_0 = (10n+9)\beta_0$ works in Proposition 4.1 and in the comments that follow it. Since $|\{y \in Q_0 : M_{0,s,Q_0}^\# f(y) > \beta_0\}| = 0$ we get a (first-generation) family $\{Q_j^1\}$ of nonoverlapping subcubes of Q_0 so that

- (1) $\delta_0 < |m_f(t, Q_j^1)| \leq \delta_0 + 10n\beta_0$ for all j ,
- (2) $|f(x)| \leq \delta_0$ for a.e. $x \in Q_0 \setminus \bigcup_j Q_j^1$, and
- (3) $\sum_j |Q_j^1| \leq s|Q_0|$.

Now we fix one cube Q_j^1 of this family, which for simplicity we denote Q^1 , and define $g = f - m_f(t, Q^1)$. Note that $m_g(t, Q^1) = 0$ and $g - m_g(t, Q) = f - m_f(t, Q)$ for all $Q \subset Q^1$. Then as above we have that $m_{|g-m_g(t,Q)|}(t, Q) \leq 2\phi(|Q|)$ for all $Q \subset Q^1$, and thus $\|M_{0,s,Q^1}^\# g\|_{L^\infty(Q^1)} \leq 2\phi(|Q|) = 2\phi(|Q_0|/2^n)$.

We pick then (first-generation) parameters $\beta_1 = 2\phi(|Q_0|/2^n)$ and $\delta_1 = (10n+9)\beta_1$, which gives $|\{y \in Q^1 : M_{0,s,Q^1}^\# g(y) > \beta_1\}| = 0$. As before we get a (second-generation) nonoverlapping family $\{Q_j^2\} \subset Q^1$ so that

- (1) $\delta_1 < |m_g(t, Q_j^2)| \leq \delta_1 + 10n\beta_1$ for all j ,
- (2) $|g(x)| \leq \delta_1$ for a.e. $x \in Q^1 \setminus \bigcup_j Q_j^2$, and
- (3) $\sum_j |Q_j^2| \leq s|Q^1|$.

We can keep control of the cubes we are gathering and f . Indeed, clearly

$$\sum_j |Q_j^2| \leq s \sum_k |Q_k^1| \leq s^2 |Q_0|.$$

And as for f we have that for a.e. $x \in Q^1 \setminus \bigcup_j Q_j^2$,

$$\begin{aligned} |f(x)| &\leq |f(x) - m_f(t, Q^1)| + |m_f(t, Q^1)| = |g(x)| + |m_f(t, Q^1)| \\ &\leq \delta_1 + \delta_0 + 10n\beta_0 \leq (20n + 9)(\beta_0 + \beta_1). \end{aligned}$$

Continuing in this fashion, the computation becomes clear: Having selected the $(k-1)$ st generation of subcubes $\{Q^{k-1}\}$, we then select a k th generation of subcubes so that

- (1) With $\beta_j = 2\phi(|Q_0|/2^{nj})$ and $\delta_j = (10n + 9)\beta_j$, $0 \leq j \leq k-1$, we have $|f(x)| \leq (10n + 9) \sum_{j=0}^{k-1} \delta_j + 10n \sum_{j=0}^{k-2} \beta_j \leq (20n + 9) \sum_{j=0}^{k-1} \beta_j$ for a.e. $x \in Q^{k-1} \setminus \bigcup_j Q_j^k$, and
- (2) $\sum_j |Q_j^k| \leq s^k |Q_0|$.

This is all that is needed. Suppose first that $\lim_{u \rightarrow 0^+} \Psi_{|Q_0|}(u) = \infty$. Then, for $\lambda > 2(20n + 9)\phi(|Q_0|)$, let k be the largest integer so that $2(20n + 9) \sum_{j=0}^{k-1} \phi(|Q_0|/2^{nj}) < \lambda$ and observe that by (1) above $\{y \in Q_0 : |f(y)| > \lambda\} \subset \bigcup_j Q_j^k$, and so by (2) above $|\{y \in Q_0 : |f(y)| > \lambda\}| \leq s^k |Q_0|$. Furthermore, by this choice of k we have

$$\begin{aligned} \lambda &< 2(20n + 9) \sum_{j=0}^k \phi(|Q_0|/2^{nj}) \\ &\leq \frac{2(20n + 9)}{n \ln(2)} \int_{|Q_0|/2^{kn}}^{2^n |Q_0|} \phi(u) \frac{du}{u} = c \Psi_{|Q_0|} \left(\frac{|Q_0|}{2^{kn}} \right) \end{aligned}$$

where $c = 2(20n + 9)/n \ln(2)$.

So,

$$\frac{|Q_0|}{2^{kn}} \leq \Psi_{|Q_0|}^{-1}(c_2 \lambda), \quad c_2 = \frac{n \ln(2)}{2(4 + 10n)}.$$

Finally,

$$|\{y \in Q_0 : |f(y)| > \lambda\}| \leq s^k |Q_0| \leq \frac{|Q_0|}{2^{kn}} \leq \Psi_{|Q_0|}^{-1}(c_2 \lambda),$$

as we wanted to show.

And, for $\lambda \leq 2(20n + 9)\phi(|Q_0|)$, pick c_1 so that $|Q_0| \leq c_1 \Psi_{|Q_0|}^{-1}(c_2 \lambda)$, and since $\{y \in Q_0 : |f(y)| > \lambda\} \subset Q_0$, the conclusion holds. Clearly the argument works for all $Q \subset Q_0$.

Finally, in case $\lim_{u \rightarrow 0^+} \Psi_{|Q_0|}(u) < \infty$, the above argument works for all integers k and, therefore, f is an essentially bounded function on Q_0 that satisfies (5.2). A more thorough argument shows that f is equivalent to an essentially Lipschitz function on Q_0 [10]. \square

It is now straightforward to verify that if $f \in vmo_s(Q_0)$, $f \in VMO(Q_0)$. Pick ϕ such that $f \in bmo_{s,\phi,Q_0}$. Then integrating (5.2) with respect to λ it follows that for all subcubes $Q \subset Q_0$,

$$\begin{aligned} \int_Q |f(y) - m_f(1-s, Q)| dy &= \int_0^\infty |\{y \in Q : |f(y) - m_f(1-s, Q)| > \lambda\}| d\lambda \\ &\leq c_1 \int_0^\infty \Psi_{|Q|}^{-1}(c_2 \lambda / \|f\|_{s,\phi,Q_0}) d\lambda \\ &\leq c \|f\|_{s,\phi,Q_0} |Q| \phi(2^n |Q|). \end{aligned}$$

Therefore,

$$\sup_{Q \subset Q_0, |Q| \leq u} \frac{1}{|Q|} \int_Q |f(y) - f_Q| dy \leq c \phi(2^n u) \rightarrow 0$$

as $u \rightarrow 0^+$ and $f \in VMO(Q_0)$.

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